

# Non-linear metric perturbations and production of primordial black holes

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We consider the simple inflationary model with peculiarity in the form of "plateau" in the inflaton potential. We use the formalism of coarse-grained field in order to describe the production of metric perturbations  $h$  of an arbitrary amplitude, and obtain non-Gaussian probability function for such metric perturbations. We associate the spatial regions having large perturbations  $h \sim 1$  with the regions going to primordial black holes after inflation. We show that in our model the non-linear effects can lead to overproduction of the primordial black holes.

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## I. INTRODUCTION

Starting from pioneering works by Zel'dovitch and Novikov [1], and also by Hawking [2], the primordial black holes (hereafter PBH's) were subject of extensive investigations. The presence of PBH's may significantly influence on physical processes and effects in the Universe (such as nucleosynthesis, CMBR spectral distortions, or distortions of  $\gamma$ -ray background radiation) due to Hawking effect [3], PBH's may be a component of dark matter (see e.g. [4], [5]). The formation of PBH's is determined by small scale, but large amplitude inhomogeneities in the Early Universe, and the processes of PBH's formation, evolution and decay link the physical conditions of Early Universe with conditions in the radiation-dominated epoch and present-day cosmology. Even the very absence of PBH's may significantly constraint the models of the beginning of cosmological evolution.

Usually the processes of PBH's formation are associated with production of the scalar mode of perturbations during inflation (see e.g. [5-9]) or phase transitions in the Early Universe [10]. In this paper we are going to discuss the first possibility, which allows to use the powerful and well-elaborated theory of instability of the expanding Universe for analysis of conditions, under which PBH's can form.

The theory of generation of adiabatic perturbations during inflation started from pioneering papers [11-13]. It was established that the RMS-amplitude of metric perturbations  $\delta_{rms}$  is connected with the parameters of inflationary theory by means of relation

$$\delta_{rms} = \frac{1}{2\pi} \frac{H^2}{|\dot{\phi}|}, \quad 1$$

where  $H$  is the Hubble parameter,  $\dot{\phi}$  is the velocity of the field, evolving during inflation. To get PBH's abundance in an observable amount, one should have  $\delta_{rms} \sim 10^{-2} - 10^{-1}$  (see, e. g. [14]). On the other hand COBE CMBR data, as well as analysis of Large-Scale Structure formation constraint the amplitude of perturbations  $\delta_{rms} \sim 10^{-5}$  at super-large scales. Therefore to get PBH's one should increase the amplitude of the perturbations by a factor  $10^3 - 10^4$  at small scales. Unfortunately this cannot be reached in the simplest inflationary models, since in these models  $\delta_{rms}$  logarithmically grows with increase of scale, and one should use nonstandard models having additional power at small scales to obtain significant PBH's amount.

Recently, several models of such type were proposed. For instance, Carr and Lidsey [6] proposed toy model having blue type spectrum (the spectrum  $\delta_{rms}(k) \propto k^a$ , where  $k$  is the wavenumber, and  $a$  is the spectral index), and investigated the constraint on the spectral index  $a$  associated with possible PBH's formation in such model. Linde [15] has shown that blue type spectra can be naturally obtained in the two-field model of so-called hybrid inflation.

Another type of model having a spike in the power spectrum at some scale  $k_{bh}$  was proposed by Ivanov, Naselsky and Novikov ([5], hereafter INN)<sup>1</sup>. They considered one-field inflationary model with inflaton  $\phi$  and assumed that the potential  $V(\phi)$  has a "plateau" region at some scale  $k_{bh}$ , and has a standard form (say, power-law form) outside the "plateau" region. The field  $\phi$  slows down in the "plateau" region giving increase of the spectrum of perturbations at the scale  $k_{bh}$  according to eqn. (1). One can adjust the parameters of "plateau" region to obtain the desired increase of the spectrum, and consequently the desired PBH's amount. Garcia-Bellido *et al* [8] and also Randall *et al* [9] considered more realistic two field models having a saddle point in two-dimensional form of potential  $V(\phi, \psi)$ . Like the one-field model, the evolution of the system of fields slows down near the saddle point giving an increase of the spectrum power. Randall *et al* pointed out that such models solve several fine-tuning problems of the standard inflation, and therefore look very natural from the point of view of high energy physics. Garcia-Bellido *et al* carefully investigated the process of PBH's formation in such models (see also recent work by Yokoyama, [18]).

If the primordial black holes are not super-large, they probably collapse during the radiation dominated epoch of the evolution of the Universe. This means that the amplitude  $h_*$  of the metric inhomogeneities inside the regions going to PBH's should be of order of unity to overcome the strong pressure forces during collapse of the perturbed region [14]. These large amplitude metric inhomogeneities are assumed to be generated during inflation as rare events in the random field of the metric perturbations. Since the amplitude of the inhomogeneities  $h_*$  is rather large, the natural question appears: to what extent we can rely on the linear theory of perturbations which usually gives Gaussian probability distribution of PBH's formation?

To answer this question we can apply the formalism of coarse-grained fields (introduced by Starobinsky [19]) as an alternative approach to the linear theory that can describe large amplitude deviations of the field and the metric from the background quantities. According to this approach, the spatially inhomogeneous field  $\phi(\vec{x}, t)$  is divided into two parts: the large-scale part  $\phi_{ls}$ , which consists of the modes with physical wavelengths  $\lambda \propto ak^{-1}$  greater than some characteristic scale  $\lambda_{c-g} \geq H^{-1}$ , and the small scale part which consists of modes with  $\lambda < \lambda_{c-g}$ . During inflation, the physical wavelengths are stretched, and new perturbations are added to  $\phi_{ls}$ . This effect may be considered as a new random force  $f(t)$  in the equation of motion of the field  $\phi_{ls}$ , and usually the dynamics of  $\phi_{ls}$  is described in terms of diffusion equation for probability density  $\Psi(\phi_{ls}, t)$ . This equation was subject of a number of works in connection with problems of Quantum Gravity and Large-Scale Structure formation. Recently, it was pointed out, that this equation can be employed for calculations of the probability to find large amplitude peaks in the random distribution of field  $\phi_{ls}$ , and it was mentioned that such approach can be applied to the problem of PBH's formation [20].

Here we would like to note that when studying the effects originating after the end of inflation, such as PBH's formation, one should use the large scale part of metric instead of large scale part of field. Contrary to the field  $\phi_{ls}$ , the large scale part of the metric, namely the "inhomogeneous scale factor  $a_{ls}(\vec{x})$ " (see eqns. (23 – 24) for exact definition) is the quantity conserving during the evolution outside the horizon, and this property allows to connect the physical conditions during inflation with the physical conditions during radiation-dominated epoch, when PBH's are formed. Moreover, the criterion for PBH's formation can be directly formulated in terms of  $a_{ls}(\vec{x})$  ( Refs. [21], [22]). Therefore, the calculation of  $a_{ls}(\vec{x})$  gives a tool to describe quantitatively the generation of non-linear metric perturbations, and the evolution of these perturbations into PBH's.

In this paper we calculate the probability distribution function  $\mathcal{P}(a_{ls}(\vec{x}))$  in the model with almost flat region in the inflaton potential. The main idea of our calculations has already been applied in the models of so-called stochastic inflation (see, e. g. [23] and references therein), and is very simple. When the field  $\phi_{ls}$  evolves inside the plateau region it slows down, and the random kicks (described by the force  $f(t)$ ) significantly influence on its evolution. So, the trajectory of the field inside the plateau region becomes stochastic, and the time  $\Delta t$  that the field spends on the plateau, depends on the realization of the stochastic process. The total increase of the scale factor  $a_{ls}$  during the field evolution on the plateau, is obviously determined by  $\Delta t$ :  $a_{ls} \propto e^{H\Delta t}$ . Since different regions of the Universe separated by distances greater than  $H^{-1}$  evolve independently, the increase of  $a_{ls}$  corresponding to different regions is determined by different realizations of the random process. Thus the scale factor  $a_{ls}$  varies from one region to another after the field passes the plateau, that is the quantum effects generate the coordinate dependence of the scale factor. The shape of function  $a_{ls}(\vec{x})$  is conserved during the subsequent evolution of the Universe until the scale of inhomogeneity crosses horizon at the second time. At that time, in the regions with significant contrast of  $a_{ls}(\vec{x})$  the primordial black holes are formed.

Using the approach described above we calculate the probability distribution function  $\mathcal{P}(a_{ls}(\vec{x}))$ . With help of a simple criterion of PBH's formation we relate  $\mathcal{P}(a_{ls}(\vec{x}))$  to the probability of PBH's formation. We show that in our

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<sup>1</sup>See also the papers by Hodges and Blumenthal, Hodges *et al*[16] and Kates *et al* [17], who employed similar models in contest of Large-Scale Structure formation theory

case the non-linear effects over-produce PBH's <sup>2</sup>. Although this result is very important qualitatively, it does not significantly change the estimate based on the linear theory.

We use the simple one-field model, proposed by INN (see also Refs. [24], [25]). Due to simplicity of this model the bulk of our results are obtained analytically. We hope that our approach provides a reasonable approximation to the case of more complicated two-field models. We are going to discuss these models in our future work.

The paper is organized as following. We introduce our model and discuss the classical dynamics of the metric and field in Section 2. In Section 3 we obtain an expression for  $\mathcal{P}(a_{ls}(\vec{x}))$ . We consider the role of non-linear effects on the statistics of PBH's production in Section 4. We summarize our conclusions and discuss applicability of our approach in Section 5.

## II. THE DYNAMICS OF CLASSICAL MODEL

In this Section we consider the classical dynamics of spatially homogeneous parts of metric and field in the simplest inflationary model with a single scalar field (inflaton) and with the peculiarity in the inflaton potential. In this case the system of dynamical equations contains only two dynamical variables -scale factor  $a(t)$  and spatially homogeneous part  $\phi_0(t)$  of the field  $\phi$ , and reduces to the Hamiltonian constraint equation

$$H^2 = \frac{8\pi}{3}(V(\phi_0) + \frac{\dot{\phi}_0^2}{2}), \quad 2$$

and to the equation of motion for field  $\phi_0$

$$\ddot{\phi}_0 + 3H\dot{\phi}_0 + \frac{\partial}{\partial\phi}V(\phi_0) = 0, \quad 3$$

where  $H = \frac{\dot{a}}{a}$ , and another symbols have their usual meaning. We hereafter use the natural system of units.

We assume that the effective potential  $V(\phi)$  has a small almost flat region ('plateau') between some characteristic values of field  $\phi_1$  and  $\phi_2$ . The potential is also assumed to be proportional to  $\phi^4$  outside the 'plateau' region

$$V(\phi) = \frac{\lambda\phi^4}{4} \quad 4$$

at  $\phi < \phi_1$

$$V(\phi) = V(\phi_1) + A(\phi - \phi_1) \quad 5$$

at  $\phi_1 < \phi < \phi_2$ , and

$$V(\phi) = \frac{\tilde{\lambda}\phi^4}{4} \quad 6$$

at  $\phi > \phi_2$ . Here  $V(\phi_1) = \frac{\lambda\phi_1^4}{4}$ ,  $\tilde{\lambda} = \lambda(\frac{\phi_1}{\phi_2})^4 + \frac{4A(\phi_2 - \phi_1)}{\phi_1^4}$ . As we will see below the size of the flat region is very small  $\Delta\phi = \phi_2 - \phi_1 \ll \phi$ ,  $\frac{A(\phi_2 - \phi_1)}{V(\phi_1)} \ll 1$  so we can set  $\lambda \approx \tilde{\lambda}$ . At sufficiently large values of  $\phi_0 > 1$  the kinetic term in the equation (2) is negligible in comparison with the potential term

$$\frac{\dot{\phi}_0^2}{2} \ll V(\phi_0), \quad 7$$

and the equation (2) reduces to an algebraical relation between  $H$  and  $\phi_0$  (so-called slow-roll approximation)

$$H = \sqrt{\frac{8\pi}{3}V(\phi_0)}. \quad 8$$

From the equation (8) it follows that the Universe expands quasi-exponentially ( $H \approx const$  and  $a \propto e^{Ht}$ ) at  $\phi_0 > 1$ .

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<sup>2</sup>Note that this result differs from that obtained in Ref. [20].

It can also be easily shown that outside the plateau region the field moves with large friction at  $\phi_0 > 1$ , so

$$|\ddot{\phi}_0| \ll |3H\dot{\phi}_0|. \quad 9$$

The friction dominated condition (9) helps to simplify the integration of the system (2 – 3). Integrating the eqns. (2 – 3) with help of inequalities (7), (9) at  $\phi_0 > \phi_2$ , we have

$$\phi_0(t) = \tilde{\phi}_0 \exp\left(-\sqrt{\frac{\tilde{\lambda}}{6\pi}}t\right), \quad 10$$

and

$$a(\tilde{\phi}_0) = a_0 \exp(N(\tilde{\phi}_0) - N(\phi_0)), \quad 11$$

where  $\tilde{\phi}_0$  and  $a_0$  are some initial values of the field and scale factor.

$$N(\phi_0) = \int_{\phi_2}^{\phi_0} H dt = \pi(\phi_0^2 - \phi_2^2) \quad 12$$

is the number of e-folds of the scale factor during the field rolling down starting from some initial value of  $\phi$  and down to the field  $\phi_2$ . The similar formulae hold at  $\phi_{end} < \phi_0 < \phi_1$

$$\phi_0(t) = \phi_1 \exp\left(-\sqrt{\frac{\lambda}{6\pi}}(t - t_1)\right), \quad 13$$

$$a(\phi_0) = a_1 \exp(N_{end}(\phi_1) - N_{end}(\phi)), \quad 14$$

where  $\phi_0(t_1) = \phi_1$  and  $a_1 = a(t_1)$ , and  $N_{end}(\phi_0)$  is the number of e-folds up to the end of inflation:  $N_{end}(\phi_0) = \pi(\phi_0^2 - \phi_{end}^2)$ , where we assume that inflation ends at standard (for  $\lambda\phi^4$  theory) value of  $\phi_{end} = \frac{1}{\sqrt{2\pi}}$ . Note, that  $N_{end}(\phi_1)$  should be rather large. For example, to get a feature in the spectrum at scales, corresponding to the solar mass, we should have  $N_{end}(\phi_1) \sim 50 - 60$ . Therefore, the value of  $\phi_1$  should be greater than unity ( $\phi_1 \sim 4.5$  for  $N_{end}(\phi_1) \sim 60$ ).

Now let us consider the dynamics of inflaton in the "plateau" region  $\phi_1 < \phi_0 < \phi_2$ . In this region the equation (3) is simplified to

$$\ddot{\phi}_0 + 3H_0\dot{\phi}_0 + A = 0, \quad 15$$

where  $H_0 = \sqrt{\frac{8\pi}{3}}V_0$ . The solution of eqn. (15) can be written as

$$\phi_0 = \phi_2 + \frac{1}{3H_0}\dot{\phi}_{in}(1 - e^{-3H_0t}) - \frac{At}{3H_0} = \phi_2 - \frac{1}{6\pi\phi_2}(1 - e^{-3H_0t}) - \frac{At}{3H_0}, \quad 16$$

and for the field velocity we have

$$\dot{\phi}_0 = \dot{\phi}_{in}e^{-3H_0t} - \frac{A}{3H_0}, \quad 17$$

where  $\dot{\phi}_{in} = \dot{\phi}_0|_{\phi_0=\phi_2} = -\frac{1}{3H_0}\frac{\partial}{\partial\phi}V(\phi_2) = -\sqrt{\frac{\tilde{\lambda}}{6\pi}}\phi_2$  is the field velocity at the moment  $t=0$  of entrance of the field in the "plateau" region. The second term in the eqn. (16) and the first term in the eqn. (17) are due to inertial influence of initial velocity  $\dot{\phi}_{in}$ , and the last terms in the both equations are due to nonzero slope of potential in the plateau region. The evolution of the field in the plateau region can be divided into two stages. At first stage the field evolves mainly due to inertial term, and velocity exponentially decreases with time. After some characteristic time  $t_*$  the nonzero slope of potential  $A$  starts to determine the evolution, the velocity tends to the constant value  $\dot{\phi}_{fd} = -\frac{A}{3H_0}$ , and the field amplitude starts to decrease linearly with time. The time  $t_*$  can be estimated by equating the inertial and potential terms in the eqn. (16), and is determined by the condition  $3H_0t_*e^{3H_0t_*} = \frac{B}{A}$ , where  $B = \frac{\partial}{\partial\phi}V(\phi_0 = \phi_1)$ .

As we discussed in Introduction, the spectrum amplitude is inversely proportional to the field velocity ( $\delta_{rms} \approx \frac{1}{2\pi} \frac{H^2}{|\dot{\phi}|}$ ), therefore we need to slow down the velocity approximately by  $\sim 10^3 - 10^4$  times to get the increase of the spectrum

amplitude from the initial value  $\delta_{rms}(in) = \frac{1}{2\pi} \frac{H^3}{B} \sim 10^{-5}$  up to the typical for PBH production  $\delta_{rms} \sim 10^{-2} - 10^{-1}$ . For that, we should fix the "amplification" parameter  $\alpha = \frac{B}{A} \sim 10^3 - 10^4$ .

Our model has two possible limiting variants depending on the relation between the time  $t_c$  of the crossing of plateau region by the field  $\phi_0$  ( $\phi_0(t_c) = \phi_1$ ) and  $t_*$ . If  $t_c \approx t_*$  the field crosses the plateau mainly due to inertia. In this case the parameter  $\alpha$  determines the number of e-folds during plateau crossing  $\delta N \approx H_0 t_c \approx \frac{1}{3} \ln \alpha \approx 2.3$ , and therefore the width of produced bump in the spectrum remains small and fixed. The model of similar type was discussed by INN. Here we consider another possible case  $t_c > t_*$ , where the field spends some time on the plateau, evolving in the friction-dominated approximation. In this case the width of the spectrum is determined by the value of  $t_c$ , which is the free parameter of our model. Instead of  $t_c$  we will parameterize our model by the quantity  $\gamma$ -the ratio of wave numbers, corresponding to the fields  $\phi_1, \phi_2$ , respectively,  $t_c = H_0^{-1} \ln \gamma$ . The parameter  $\gamma$  cannot be too small  $\gamma > \alpha^{1/3}$  and we take  $\gamma \approx 10^3$  in the estimations. If  $\gamma$  is not extremely large  $\ln \gamma \ll N(\phi_1)$ , the size of plateau  $\Delta\phi_0 = \phi_2 - \phi_1$  is of order of typical size  $\Delta\phi_* = \frac{B}{9H^2}$ . The typical relative size of plateau is very small

$$\frac{\Delta\phi_0}{\phi_0} = \frac{1}{6\pi\phi_0^2} \approx \frac{1}{6N(\phi_1)} \approx 0.003. \quad 18$$

Thus, the correction due to the presence of plateau practically does not influence on the dynamics of the field outside plateau region and we can set  $\lambda = \tilde{\lambda}$ . On the other hand, the size of plateau is much greater than  $H_0$  - the typical size of quantum fluctuations,  $\Delta\phi_* = \frac{H_0}{6\pi\delta_{rms}(in)} \sim 10^5 H_0$ .

Typically, the estimate  $\frac{\Delta\phi_0}{\phi_0} \ll 1$  holds for arbitrary power-law potentials  $V(\phi) \propto \phi^p$  provided power  $p$  is not very large. However the opposite limiting case is also possible. For example, Bullock and Primack [20] proposed the potential of the form

$$V(\phi) = \lambda_{bp}(1 + \arctan(\phi)), \quad \phi > 0$$

$$V(\phi) = \lambda_{bp}(1 + 4 * 10^{33} \phi^{21}), \quad \phi < 0 \quad 19$$

where the constant  $\lambda_{bp} = 6 * 10^{-10}$  is chosen to normalize the large-scale part of spectrum to the RMS-amplitude  $\approx 3 * 10^{-5}$ . The flat region in this potential starts from  $\phi = 0$  and ends at  $\phi = -1.23 * 10^{-2}$ , and inflation ends itself at  $\phi = \phi_{end} = -1.55 * 10^{-2}$ . It was mentioned by Bullock & Primack that this potential leads to strongly non-Gaussian statistics of field perturbations.

### III. NON-LINEAR METRIC PERTURBATIONS FROM THE QUANTUM DYNAMICS OF COARSE-GRAINED FIELD

It is well known that there are two equivalent ways to describe inhomogeneous Universe. The first way is to consider inhomogeneities as a small corrections to the homogeneous space-time and study them in the frameworks of linear theory of perturbations. Another approach splits the metric and the field into large-scale part (coarse-grained over some scale greater than horizon scale), and small-scale part. During inflation, the dynamical equations for coarse-grained field  $\phi_{ls}$  and coarse-grained scale-factor  $a_{ls}$  are equivalent to eqns. (3, 8) provided the quantum effects are switched off. The quantum effects continuously produce new inhomogeneities of random amplitude with scales greater than the scale of coarse-graining. These inhomogeneities should be added to  $\phi_{ls}$  and  $a_{ls}$  and effectively this leads to the presence of stochastic force term in the equations of motion. Therefore, the dynamics of coarse-grained variables can be described in terms of the distribution functions of  $\phi_{ls}$  and  $a_{ls}$ , and in principal these distribution functions can provide the same information as the power spectrum of perturbations, and furthermore the coarse-grained formalism gives a tool for description of the metric perturbations with amplitude, greater than 1.

The effective dynamical equation for the field  $\phi_{ls}$  has the form [19]<sup>3</sup>

$$\ddot{\phi}_{ls} + 3H_{ls}\dot{\phi}_{ls} + \frac{\partial}{\partial\phi}V(\phi_{ls}) = D^{1/2}f(t), \quad 20$$

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<sup>3</sup>See also recent papers [26] and references therein.

where  $D = \frac{9H_{ls}^5}{(2\pi)^2}$ , and  $f(t)$  is delta-correlated random force,  $\langle f(t_1)f(t_2) \rangle = \delta(t_1 - t_2)$ . The equation for coarse-grained scale factor  $a_{ls}$  remains unchanged

$$H_{ls} = \sqrt{\frac{8\pi}{3}V(\phi_{ls})}. \quad 21$$

The solution of the set of eqns. (20,21) is extremely difficult problem, and can be done under some additional simplifying assumptions. For example if we choose the featureless potential, and consider the friction-dominated solutions of the eqn. (20), we can obtain the solutions describing self-reproduced inflationary Universe (provided the stochastic term in (20) dominates over potential term, see for example Linde [23]). In our case we cannot use the friction-dominated condition in the beginning of the field evolution inside the plateau region. However we can adopt another simplifying assumptions: first we can set  $H_{ls} = H_0 = \text{const}$  inside and near the plateau region, and second, we can omit the stochastic term in the eqn. (20) outside the plateau region, assuming the field moves along the classical trajectory there. Under these assumptions the statistics of the scale factor  $a_{ls}$  is totally determined by the time  $\Delta t$  that field  $\phi_{ls}$  spends in the plateau region

$$\Delta N = \ln(a_{out}/a_{in}) = H_0 \Delta t, \quad 22$$

where  $a_{in}$  is the value of scale factor at the time  $t = 0$  of entrance of the field in the plateau region, and  $a_{out}$  corresponds to the moment  $\Delta t$ , when the field leaves the plateau region. To see that let us consider the evolution of the scale factor  $a_{ls}$  in the comoving coordinate system. Outside the horizon the hypersurfaces of constant comoving time  $t_{com}$  practically coincide with hypersurfaces of constant energy density  $\epsilon = \text{const}$ . On the other hand, the field  $\phi_{ls}$  evolves slowly during inflation and hypersurfaces of constant energy density are close to hypersurfaces  $\phi_{ls} = \text{const}$ , and therefore we can put  $a_{ls}(t_{com}) = a_{ls}(\phi_{ls})$ . After the field passes the plateau region, the evolution of  $a_{ls}(\phi_{ls})$  can be described by the standard expression (14), so we have

$$a_{ls}(\phi_{ls}) = a_{in} \exp(\pi(\phi_1^2 - \phi_{ls}^2) + \Delta N), \quad 23$$

where  $\Delta N$  is nearly constant inside of coarse-grained regions with comoving scale  $\lambda_{c-g} \approx a_{out}H_0^{-1}$ , but changes from one region to another. Thus, the metric outside horizon has the quasi-isotropic form

$$ds^2 = dt^2 - a_{ls}^2(\phi_0)a_{ls}(\vec{x})\delta_j^i dx_i dx^j, \quad 24$$

where we represent the scale factor  $a_{ls}(\phi_{ls})$  as a multiplication of two factors:  $a(\phi_0)$  and  $a_{ls}(\vec{x}) \equiv e^{\Delta N}$ . Here  $a_{ls}(\phi_0)$  and  $\phi_0(t)$  are determined by the classical equations (13), (14), and the spatial coordinates  $\vec{x}$  are coarse-grained over the regions with scale  $\lambda_{c-g}$ . To estimate the change of metric from one region to another quantitatively, we introduce the definition of non-linear metric perturbation

$$h \equiv \frac{a_{ls}(\phi_{ls}) - a(\phi_0)}{a(\phi_0)} = \exp H_0(\Delta t - t_c) - 1 \quad 25$$

(remind, that  $t_c = H_0^{-1} \ln \gamma$  is the time which the field spends in the plateau region moving along the classical trajectory when the stochastic term in (20) is switched off). Note, that in the limit of small  $h \ll 1$ , the metric assumes the form

$$ds^2 = dt^2 - a^2(\phi_0)(1 + 2h(\vec{x}))\delta_j^i dx_i dx^j, \quad 26$$

and the definition (25) is reduced to the standard expression for growing mode of adiabatic perturbation outside the horizon. Namely, in this case  $h$  reduces to gauge independent quantities, introduced by a number of authors [11-13], [27] up to a constant factor. The variables (25,26) do not depend on time outside the horizon. Therefore, using of these variables is very convenient to match the perturbations, generated during inflation with the perturbations, crossing horizon at the normal stage of the Universe evolution. As one can see from (25) the metric perturbations are determined by stochastic variable  $\Delta t$  and the distribution of  $\Delta t$  must follow from the solution of eqn. (20). Note, that the definition of non-linear metric perturbations should be taken with a caution. In principal, one can use another definition relating to (25) by some non-linear transformation, and having the same limit (26) in the case of small  $h$ . For example, Bond and Salopek [28] used the quantity  $\tilde{h} = \ln(\frac{a_{ls}(\phi_{ls})}{a(\phi_0)})$  to define non-linear metric perturbations. However, the criterion for PBH's formation can be directly expressed in terms of the quantity (25) (see next Section), and therefore this quantity is the most natural variable for our purposes.

Although the assumption of constant  $H_0$  greatly simplifies the problem it still remains rather complicated for a simple analytical treatment <sup>4</sup>

For further progress we have to make some additional assumptions. We will consider below the plateau region of sufficiently large size. For this case the field approaches to the end of plateau in the friction-dominated approximation, which greatly simplifies the treatment of diffusion processes. To estimate the relevance of friction-dominated approximation we should compare the time  $t_c$  and the time  $t_* \sim \ln(\alpha)$  of the decay of the inertial term  $\ddot{\phi}$  in the eqns. (15 – 17, 20). If  $t_c > t_*$  and therefore  $\gamma \gg \alpha^{1/3}$ , the inertial term in these equations can be neglected at  $t_* < t < t_c$ . In this regime the solution of the classical equation of motion (15) has the form

$$\phi_0(\tau) \approx \phi_2 - a\tau, \quad (27)$$

and the equation (20) becomes

$$\frac{d\phi_{ls}}{d\tau} + a = d^{1/2}f(\tau), \quad (28)$$

where  $\beta = 3H_0$ , and we introduce the dimensionless time  $\tau = \beta t$ ,  $a = A/\beta^2$  and  $d = \frac{D}{2\beta^3} = \frac{H_0^2}{24\pi^2}$ . The stochastic equation (28) is associated with simple diffusion type equation, describing the evolution of positions probability distribution  $\Psi(\tau, \phi)$

$$\frac{\partial \Psi}{\partial \tau} = d \frac{\partial^2}{\partial \phi^2} \Psi + a \frac{\partial}{\partial \phi} \Psi, \quad (29)$$

Now we assume that the distribution  $\Psi$  is not spread out sufficiently before  $\tau_* = \beta t_*$  and take  $\delta$ -distributed  $\Psi$  function at the moment  $\tau = \tau_*$  as the initial condition for our problem

$$\Psi(\tau_*) = \delta(\phi_{ls} - \phi_*), \quad (30)$$

where  $\phi_* = \Delta\phi - a\tau_*$  is the value of field corresponding to the beginning of "friction-dominated" part of plateau region <sup>5</sup>.

Together with initial condition (30) we should specify the boundary condition at  $\phi_{ls} = \phi_1$ . This condition depends on the form of the transition layer between the plateau region and the part of potential with steep slope  $\frac{\partial}{\partial \phi} V(\phi) = B$ . We assume this transition to be sharp, and therefore set the condition of absorbing wall at the downstream point  $\phi_{ls} = \phi_1$

$$\Psi(\phi_1, \tau) = 0, \quad (31)$$

Note, that this boundary condition was used by Aryal and Vilenkin [32] for analysis of stochastic inflation in the theory with top-hat potentials. In that paper it was shown that the more reasonable smooth transitions between the flat and steep regions of the potential are unlikely to modify significantly the resulting distribution.

In our case the probability density  $P(\tau)$  of time  $\tau$  relates to the solution of eqn. (27) as

$$\mathcal{P}(\tau) = S|_{\phi_{ls}=\phi_1} = d \frac{\partial}{\partial \phi} \Psi, \quad (32)$$

where we define by  $S$  the probability current  $S = d \frac{\partial}{\partial \phi} \Psi + a\Psi$ . The conservation of the probability current allows to estimate the correction term to eqn. (32) due to nonzero  $\Psi(\phi_1)$ . Assuming that field moves along the classical trajectory after  $\phi_{ls} = \phi_1$ , we have  $S(-\phi_1) \approx B/\beta^2 \Psi \approx S(+\phi_1) \approx d \frac{\partial}{\partial \phi} \Psi$ . Therefore the correction to the expression (32) is  $\frac{\beta^2 a}{B} = \alpha^{-1} \sim 10^{-3} - 10^{-4}$  times smaller than the leading term.

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<sup>4</sup>In this case our problem is reduced to the first-passage problem for one-dimensional Fokker-Plank (Kramers) equation, associated with eqn. (20) [29]. The general solution of this problem demands too much formalism [30], and is not considered here. Note, however that the simple asymptotic estimates are still possible in this case [30], [31].

<sup>5</sup>The estimates show that the characteristic width of  $\Psi(\tau_*)$  is of order of  $H$  and much less than the size of "friction-dominated" region  $\phi_* - \phi_1$ .

The conditions (30,31) determine the solution of eqn. (29). This solution can be found by standard methods of the theory of diffusion equations (see, e. g. Ref. [33]), and in our case has the form

$$\Psi(\phi, \tau) = \frac{1}{\sqrt{4\pi d(\tau - \tau_*)}} \exp \left\{ -\frac{1}{4d(\tau - \tau_*)} (\phi - \phi_* + a(\tau - \tau_*)^2) \right\} \\ (1 - \exp \left\{ -\frac{1}{d(\tau - \tau_*)} (\phi - \phi_1)(\phi_* - \phi_1) \right\}), \quad (33)$$

Substituting (33) to the equation (32) we find the explicit expression for  $\mathcal{P}(\tau)$

$$\mathcal{P}(\tau) = \frac{1}{\sqrt{(4\pi d(\tau - \tau_*)}} \left( \frac{\phi_* - \phi_1}{\tau - \tau_*} \right) \exp \left\{ -\frac{1}{4d(\tau - \tau_*)} (\phi_1 - \phi_* + a(\tau - \tau_*)^2) \right\}. \quad (34)$$

The expression for probability distribution of metric can be readily obtained from (34). Using the eqn. (22 – 25) to express the time  $\tau$  in terms of  $h$ , taking into account the eqn (27) and the definitions of  $a$ ,  $d$ , and assuming  $a > 0$ , we obtain

$$\mathcal{P}(h) = \frac{1}{\sqrt{2\pi\delta_{pl}^2}} \frac{N_{cl}}{N_{st}^{3/2}} \frac{dN_{st}}{dh} \exp \left\{ -\frac{(N_{st} - N_{cl})^2}{2\delta_{pl}^2 N_{st}} \right\}, \quad (35)$$

where  $\delta_{pl} = \frac{3H_0^3}{2\pi A} = \alpha\delta_{rms}(in)$  is the standard metric amplitude calculated for the plateau parameters, and

$$N_{cl} = \ln \gamma - \tau_*/3, \quad N_{st} = \ln(1 + h) + N_{cl} \quad (36)$$

are the numbers of e-folds for the classical path  $\phi_0(t)$  and for a random path  $\phi_{ls}(t)$ , which start at  $\phi_* = \phi(t_*)$  and end at  $\phi_1$ .

When the perturbations are small  $N_{st} - N_{cl} \approx h \ll 1$ , the distribution (35) has the standard Gaussian form

$$\mathcal{P}(h) = P_G(h) = \frac{1}{\sqrt{2\pi\delta_{pl}^2 N_{cl}}} \exp \left\{ -\frac{h^2}{2\delta_{pl}^2 N_{cl}} \right\}, \quad (37)$$

and in the opposite case of very large metric perturbations  $h \gg 1$  and  $N_{st} \sim \ln h > N_{cl}$  the distribution  $\mathcal{P}(h)$  deviates sharply from the Gaussian law and has the power-law form

$$\mathcal{P}(h) \propto h^{3/2 + \delta_{pl}^{-2}/4}, \quad (38)$$

As seen from eqns. (35 – 38), the non-Gaussian effects over-produce the metric perturbations of high amplitude in our model. To understand this fact, let us discuss the origin of non-Gaussian effects in our model. There are two sources for such effects. First, note that the "effective dispersion"  $\sigma_{eff}^2 = \delta_{pl}^2 N_{st}$  in eqn. (35) depends itself on the value of the stochastic variable  $N_{st}$ . Qualitatively, it can be explained as follows. In the linear theory the dispersion  $\sigma^2 = \delta_{pl}^2 N_{cl}$  is proportional to the time spent by the classical background field  $\phi_0$  on the plateau. In non-linear theory the coarse-grained field  $\phi_{ls}(t)$  plays the role of background field, and therefore the distribution of the family of neighboring to  $\phi = \phi_{ls}(t)$  paths should be described in terms of the probability distribution with dispersion  $\sigma_{eff}^2$ , which is proportional to the time spent by field  $\phi_{ls}$  on the plateau. Second, the amplitude of large metric perturbations  $h$  depends on  $N_{st}$  exponentially ( $h \sim e^{N_{ls}}$ ), so order of magnitude increase of  $N_{st}$  leads to exponential increase of  $h$ . Obviously, these two effects increase the probability of large amplitude metric perturbations.

#### IV. PROBABILITY OF BLACK HOLES FORMATION

Although the distribution (35) provides very important information about the geometry of spatial part of metric outside horizon, it cannot be directly applied to the estimates of PBH's formation. Indeed, the distribution (35) is formed by the field inhomogeneities with wave-numbers  $k$  in the range ( $\Delta k = [k_{min} \approx a_{in} H_0 < k < k_{max} \approx a_{out} H_0]$ ). The process of PBH formation is determined mainly by the field modes with wave-numbers ( $\delta k \approx k_{bh} \ll \Delta k$ ), where  $k_{bh}$  is the typical PBH wavenumber. The modes with  $k < k_{bh}$  compose the large-scale background part of metric at the moment of PBH formation, and do not influence on the formation of PBH's significantly. The modes with



$k > k_{bh}$  lead to high-frequency modulation of the perturbation with  $k \sim k_{bh}$ , which is also unimportant, provided the mode with  $k \sim k_{bh}$  crosses the horizon second time at the radiation-dominated epoch. Therefore, in order to obtain the probability of PBH's formation, we should subtract the contribution of the large-scale and small-scale metric perturbations.

In general it is very difficult to separate the perturbations of a given scale in the frameworks of non-linear approach. However, we can estimate the probability density of the perturbations, corresponding to the smallest scale  $k_{bh} \approx a_{out}H_0$ <sup>6</sup>. For that we simply put  $N_{cl} = 1$  in eqns. (35, 36), assuming that the random process starts when the mode with wavenumber  $k_1 = e^{-1}a_{out}H_0$  crosses horizon. This procedure automatically subtracts the large-scale contribution of modes with  $k < k_1$ . The small-scale contribution is also absent due to our absorbing boundary condition. We have

$$\mathcal{P}(h) = \frac{1}{\sqrt{2\pi\delta_{pl}^2}} \frac{1}{(x+1)^{3/2}} \exp\left\{-\frac{x^2}{2\delta_{pl}^2(x+1)}\right\} \quad (39)$$

from the eqn. (35), where  $x = \ln(1+h)$ , and in the limit of small  $h$  we obtain again the Gaussian distribution

$$\mathcal{P}(h) \approx \mathcal{P}_G(h) = \frac{1}{\sqrt{2\pi\delta_{pl}^2}} \exp\left\{-\frac{h^2}{2\delta_{pl}^2}\right\}. \quad (40)$$

The distribution (39) has nonzero first momentum  $M_1 = \int_{-1}^{\infty} dh h \mathcal{P}(h) = \frac{3}{2}\delta_{pl}$  (the lower limit of integration should be -1, since the metric perturbations with  $h < -1$  are cut off). The contribution of  $M_1$  should be added to the background part of metric, and further we will use the renormalized metric perturbation  $h_r = h - \frac{3}{2}\delta_{pl}$  instead of  $h$ . The probability to find the metric perturbations  $h_r$  with amplitude greater than some threshold value  $h_*$   $P(h_*) = \int_{h_*}^{\infty} dh \mathcal{P}(h)$  can be estimated as

$$P(h_*) \approx \frac{1}{\sqrt{2\pi}} \left(\frac{2\delta_{pl}(x_*+1)^{1/2}}{x_*(x_*+2)}\right) \exp\left\{-\frac{x_*^2}{2\delta_{pl}^2(x_*+1)}\right\}, \quad (41)$$

where  $x_* = \ln(1 + \frac{3}{2}\delta_{pl} + h_*)$ , and we assume  $h_* \gg \delta_{pl}$ . The same quantity, but calculated for the Gaussian distribution takes the well-known form

$$P_G(h) \approx \frac{1}{\sqrt{2\pi}} \frac{\delta_{pl}}{h_*} \exp\left\{-\frac{h_*^2}{2\delta_{pl}^2}\right\}. \quad (42)$$

The observed quantities (such as, e.g. the matter density of PBH's in different cosmological epochs) can be easily expressed in terms of the probability  $P(h_*)$ , provided the mass of PBH's and some criterion for PBH's formation are fixed. In our case the criterion for PBH's formation should give the information about the threshold value  $h_*$ . Since this criterion plays very important role, let us discuss it in some details. First let us note that PBH's are formed from high amplitude peaks in the density distribution which are approximately spherically-symmetric (see e.g. Ref. [34]). It can also be easily shown that the maxima in the matter density correspond to the maxima in the function  $a_{ls}(\vec{x})$ . The form of  $a_{ls}(\vec{x})$  totally specifies the number of regions going to PBH's as well as dynamics of the collapsing regions. Therefore we formulate the criterion of PBH's formation in terms of conditions imposed on the function  $a_{ls}(\vec{x})$ .

The first criterion was formulated by Carr in his seminal paper [14]. It was shown that an over-dense region forms PBH if the density contrast at the horizon scale  $\frac{\delta\rho}{\rho}$  lies approximately within the limits  $\frac{1}{3} < \frac{\delta\rho}{\rho} < 1$ . The first part of this inequality tells that the over-dense region should stop expansion before the scale of the region crosses the sound horizon. The second part requires that the over-dense region does not collapse before crossing the causal horizon, and consequently the perturbation does not produce a closed world separated from the rest of the Universe. Then the criterion for PBH's formation was improved by Nadejin, Novikov and Polnarev [21] (hereafter NNP), and also by Biknell and Henriksen [22] with help of numerical computations. The initial condition used by NNP was chosen as a non-linear metric perturbation having the form of a part of the closed Friedman Universe matched with the spatially flat Universe through an intermediate layer of negative density perturbation. The conditions for PBH's formation depend on the size of this part (i.e. the amplitude of the perturbation), as well as on the size of the matching layer.

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<sup>6</sup>In this connection, let us note that the black holes of smallest mass should give the major contribution to the present fraction of black holes, provided PBH's spectrum is flat (Carr, 1975 [14]).

The smaller matching layer is, the larger the pressure gradients needed to prevent collapse will be. Therefore, the amplitude of the perturbation forming PBH must be greater in the case of narrow intermediate layer. In terms of our function  $a(\vec{x})$  the NNP criterion reads

$$h_* \equiv \frac{a_+}{a_-} - 1 > 0.75 - 0.9 \quad 43$$

where  $a_+$  is the value of  $a(\vec{x})$  at the maximum of the perturbation and  $a_-$  is the same quantity outside the perturbed region<sup>7</sup>. The first number on the right hand side of (43) corresponds to the matching layer of size comparable with size of the over-dense region, and the second number corresponds to the narrow matching layer. Assuming the matching layer to be sufficiently large we take  $h_* = 0.75$  as a criterion of PBH's formation.

Once the criterion is specified, we can link the desired PBH's abundance  $\beta(M_{pbh}) \approx P(h_{*pbh})$  with the parameters of our model. For instance, consider the model having the matter density of PBH's equal to the critical one (the density parameter  $\Omega_{pbh} = 1$ ). In this model we have [3], [6]

$$\beta(M) = 10^{-8} \left( \frac{M}{M_\odot} \right)^{1/2}. \quad 44$$

Equating the expression (44) to the probability function (39), we have the equation determining the amplitude  $\delta_{1pl}$  required for PBH's abundance (44) as a function of  $M_{pbh}$

$$P(h_{*pbh}, \delta_{1pl}) = \beta(M_{pbh}), \quad 45$$

and equating the expressions (42) and (44) we obtain the analogous equation for determining the reference amplitude  $\delta_{2pl}$  when the non-Gaussian effects are switched off. The solution of these equations is given in Fig. 1.

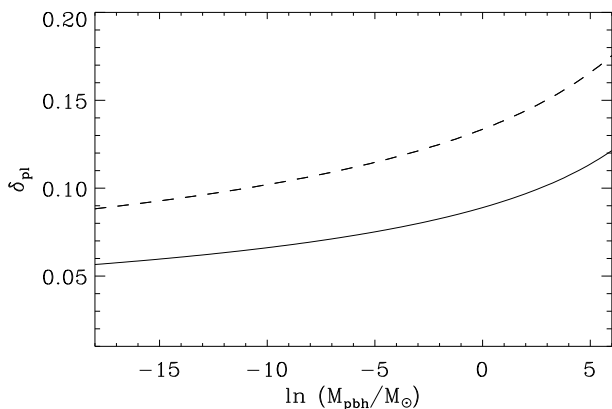


FIG. 1. We plot the dependence of plateau parameter  $\delta_{pl}$  on PBH's mass  $M_{pbh}$  assuming that the PBH's abundance is given by the eqn. (44). The solid line represents the solution of eqn. (45) (i.e we calculate  $\delta_{pl}$  taking into account the non-Gaussian effects in this case). The dashed line represents  $\delta_{pl}$  calculated in the standard Gaussian theory. The PBH's masses lie in the range:  $10^{-18} M_\odot < M_{pbh} < 10^6 M_{pbh}$ . The PBH's of the mass  $10^{-18} M_\odot \sim 10^{15} g$  should be evaporated at the present time. Actually, the abundance of these PBH's is limited much stronger than is assumed in our calculations.

One can see from this Fig. that the quantities  $\delta_{1pl}$  and  $\delta_{2pl}$  increase with increasing of  $M_{pbh}$  and  $\delta_{1pl}$  is always smaller than  $\delta_{2pl}$ . It means that non-Gaussian effects over-produce PBH's in our model (at least when the simple criterion (43) is used), and the slope of potential can be steeper than that required in the Gaussian case. Typically, the ratio  $\frac{\delta_{2pl}}{\delta_{1pl}}$  is about 1.5. Say, for the case of  $M_{pbh} = M_\odot$ , we have  $\delta_{1pl}(M_\odot) \approx 0.089$  and  $\delta_{2pl}(M_\odot) \approx 0.134$ . We plot the probability function  $\mathcal{P}(h)$  for  $\delta_{1pl}(M_\odot) = 0.089$  in Fig. 2.

<sup>7</sup>In the linear theory the density perturbation at horizon scale relates to the metric perturbation by  $\frac{\delta\rho}{\rho} = \frac{4}{9}h$  (see, e.g. [27]). Therefore the estimate (43) is in agreement with Carr's result.

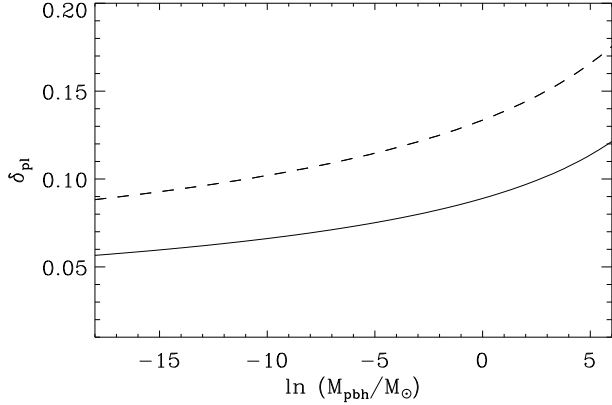


FIG. 2. The dependence of probability density  $\mathcal{P}(h)$  on the metric amplitude  $h$ . The non-Gaussian curve (solid line) is calculated with help of eqn. (39) assuming PBH's abundance  $\beta(M_\odot) \approx 10^{-8}$ . That gives  $\delta_{1pl}(M_\odot) \approx 0.089$ . The dashed line is the reference Gaussian probability density calculated for the same abundance. For that curve we have  $\delta_{2pl}(M_\odot) \approx 0.134$ . The dotted curve represents the Gaussian distribution taken with  $\delta_{1pl}(M_\odot) \approx 0.089$ . This distribution strongly under-produces PBH's, and in this case we have  $\beta \sim 10^{-17}$ .

In this Fig., we also plot the Gaussian probability function  $\mathcal{P}_G(h)$  for  $\delta_{2pl}(M_\odot) = 0.134$  (dashed line) and the same quantity for  $\delta_{1pl}(M_\odot) = 0.089$  (dotted line). Comparing the curves that correspond to the same PBH's abundance, we see that the non-Gaussian curve is flatter having larger values of  $\mathcal{P}(h)$  at large  $h$ . The values of the Gaussian curve with the same plateau parameter  $\delta_{1pl}(M_\odot)$  is smaller by many orders of magnitude than the values of the non-Gaussian curve in the case of large  $h$ .

Finally, let us note, that the non-Gaussian effects does not modify significantly the estimates based on the Gaussian theory. As we have seen, the ambiguity in the choice of the plateau slope due to these effects is about 1.5. This ambiguity seems to be less than the ambiguity in other parameters and can be obviously absorbed by a small change of the potential slope.

## V. DISCUSSION

We have demonstrated that the non-Gaussian effects related to the dynamics of the coarse-grained field (inflaton) and to the evolution of the large-scale part of metric over-produce large-amplitude inhomogeneities of the metric compared to the prediction of the Gaussian (linear) theory of perturbations. We have derived an analytical expression for non-Gaussian probability distribution for non-linear metric perturbations, and estimated the influence of non-linear effects on the probability of primordial black holes formation. We used the simple single field inflationary model with peculiarity in form of the flat region in inflaton potential  $V(\phi)$ , and power-law slope of the potential outside the peculiarity region. The key point of our approach is in the using of inhomogeneous coarse-grained metric function  $a(\vec{x})$  instead of the coarse-grained field  $\phi_{ls}$  as a basic quantity. This allows to match the physical condition of production of inhomogeneities during inflation with the "observable" quantities.

Our results can be considered as semi-qualitative only. The uncertainties come from the phenomenological character of our inflationary model as well as from the oversimplified treatment of the process of PBH's formation. The uncertainties related to the choice of parameters of inflationary model are mainly due to unknown form of the potential between the steep and flat regions, and also due to our friction-dominated assumption in the consideration of the stochastic process. These uncertainties can be eliminated with help of numerical simulations of stochastic process in more realistic models of inflation. The ambiguities concerning the criterion of PBH's formation are mainly due to the one-point treatment of this process. Actually, PBH formation is nonlocal, and dynamics of collapsing region depends strongly on the form of the spatial profile of the density perturbation (see e.g. Refs. [22], [35] for discussion of this point). The form of the spatial profile can be studied by means of n-point correlation functions of the coarse-grained metric and field. Unfortunately, the formalism of n-point correlation functions is still not elaborated (see, however the Ref. [35] for the first discussion). Note, that probably the influence of the spatial profile of the collapsing region may be taken into account by a redefinition of the threshold value  $h_*$ , and this value might be effectively less. In this case the role of the non-linear effects would be damped.

Finally we would like to note that the form of the distribution (35) does not depend explicitly on the specific parameters of our model. This allows to suppose that similar distributions can be obtained in more complicated

models, say, in two-field models proposed in Refs. [8], [9]. We are going to check this very interesting assumption in our future work.

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